# Separation of variables and exact solution of the Klein-Gordon and Dirac equations in an open universe

Víctor M. Villalba[*][†]

*Institut für Theoretische Physik. Universität Frankfurt*

*D-60054 Frankfurt am Main, Germany*

Esteban Isasi Catalá[‡]

*Departamento de Fisica, Universidad Simón Bolívar,*

*Caracas, 89000 Venezuela.*

## Abstract

We solve the Klein-Gordon and Dirac equations in an open cosmological universe with a partially horn topology in the presence of a time dependent magnetic field. Since the exact solution cannot be obtained explicitly for arbitrary time-dependence of the field, we discuss the asymptotic behavior of the solutions with the help of the relativistic Hamilton-Jacobi equation.

Typeset using REVTₑX

[†]Alexander von Humboldt Fellow

[‡]eisasi@fis.usb.ve

[*]On leave from Centro de Física, Instituto Venezolano de Investigaciones Científicas, IVIC Apdo 21827, Caracas 1020-A, Venezuela. e-mail: villalba@th.physik.uni-frankfurt.de

# I. INTRODUCTION

During the last years a large amount of observational data has been reported showing that our universe is almost isotropic and homogeneous. The study of the structure of the Cosmic Microwave Radiation leads us to conclude that the ratio of the total density to the critical density of the universe $\Omega_0$ is likely to be close to one [1–3], favoring a spatially flat Robertson-Walker metric over other topologies.

It is well known that general relativity is a local metrical theory and therefore the corresponding Einstein field equations do not fix the global topology of spacetime and consequently the universe may have compact spatial sections with a nontrivial topology [4,5], then the observational data does not rule out the possibility that our universe possesses a hyperbolic topology [4,6–8].

The study of cosmological models with nonstandard topologies is not new and goes back to the works by Zelmanov [9,10], showing that upon different coordinate transformations, spatially closed or flat sections can be transformed into hyperbolic sections and vice versa.

The line element associated with an spatially open Friedman universe has the form

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + dr^2 + \sinh^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \tag{1}$$

Making the coordinate transformation [5]

$$e^{-z} = \cosh r - \sinh r \cos \theta, \, e^{-z} x = \sin \theta \cos \phi \sinh r, \, e^{-z} y = \sin \theta \sin \phi \sinh r \tag{2}$$

the metric (1) becomes

$$a^{-2}(\eta) ds^2 = -d\eta^2 + dz^2 + e^{-2z} \left( dx^2 + dy^2 \right) \tag{3}$$

The topology is induced by identifying points periodically along $x$ and $y$ by $(x,y) = (x + b, y + h)$, where $b$ and $h$ are constant, to create a two dimensional torus. The torus is stretched by the factor $e^{-2z}$ along the $z$ axis to create a toroidal horn. The comoving proper area of the torus is $e^{-2z}bh$ and depends on location along the $z$ axis. The global topology induces global inhomogeneity as well as global anisotropy [4].

The study of quantum effects in cosmological universes with a nontrivial topology allows us a deeper understanding of the properties of different scenarios and which of them can describe our universe. In this direction the metric (3) represents a very interesting scenario in order to discuss particle production and propagation of perturbations in cosmology.

After the publication of the pioneering article by Schrödinger [11], discussing particle production in a deSitter universe, many articles have been published on the problem of quantum effects in cosmological scenarios [12–14], most of them dealing with a Robertson-Walker line element with spatially flat topology. This particularly simple line element, which is the most used in inflationary models, permits one to compute the Green function as well as to solve the relativistic wave equations [15–17].

In order to study quantum processes in curved space-times one has to fulfill a preliminary step which consists in having a description of the single-mode solution of the relativistic particles or perturbations in those background fields, i.e., exact solution of the relativistic scalar and spinor wave equations. In the literature we have at our disposal different methods for solving relativistic wave equations in curved spaces and in curvilinear coordinates; among them the method of separation of variables is one of the most widely used. [18–21].

It is the purpose of the present article to solve the Klein-Gordon and Dirac equations in the Friedman universe associated with the metric (3) in the presence of a time dependent magnetic field. In order to solve the Dirac equation we apply the algebraic method of separation of variables [20–24]. We compare the solutions with those of obtained after solving the relativistic Hamilton Jacobi equation. The article is structured as follows: In Sec. II we solve the relativistic Hamilton-Jacobi equation in an open cosmological universe with a horn topology. In Sec. III we separate variables and solve the Klein-Gordon equation. In Sec IV, using the algebraic method of separation of variables, we reduce the Dirac equation to a system of first order coupled differential equations that we solve in terms of special functions. Finally, in Sec. V we briefly discuss the results reported in this article.

## II. SOLUTION OF THE HAMILTON-JACOBI EQUATION

The covariant generalization of the Hamilton-Jacobi equation has the form [25]

$$g^{\alpha\beta}\left(\frac{\partial S}{\partial x^\alpha} - eA_\alpha\right)\left(\frac{\partial S}{\partial x^\beta} - eA_\beta\right) + M^2 = 0, \tag{4}$$

where $g^{\alpha\beta}$ is the contravariant metric, $A_\alpha$ is the vector potential and $M$ is the mass of the particle. Here and elsewhere we adopt the convention $c = \hbar = 1$.

Let us introduce an electromagnetic field associated with the vector potential

$$A^\mu = A_1(y)\delta_1^\mu, \tag{5}$$

where the index $\mu = 0$ is associated with the evolution parameter $\eta$ and $\mu = 1, 2, 3$ correspond to the space coordinates $x, y, z$ respectively. Looking at the relativistic invariants

$$\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = B^2 - E^2 = \frac{e^{4z}}{\alpha(\eta)^4}\left(\frac{dA_1(y)}{dy}\right)^2, \tag{6}$$

$$F^{\mu\nu}F_{\mu\nu}^* = 0, \tag{7}$$

and taking into account that only $F_{23}$ is different from zero, we notice that (5) corresponds to a non constant magnetic field $B$, directed along the $z$ axis, with strength.

$$B = \frac{e^{2z}}{\alpha(\eta)^2}\left|\frac{dA_1(y)}{dy}\right|, \tag{8}$$

whose value is inversely proportional to the expansion factor $\alpha(\eta)^2$.

The line element (3) is a Stäckel space [26], and the Hamilton-Jacobi equation (4) is completely separable in (3) in the presence of the vector potential (5), therefore we can look for a solution in the form

$$S = k_x + S_y(y) + S_z(z) + S_\eta(\eta). \tag{9}$$

Substituting (9) into Eq. (4) we obtain

$$\frac{(k_x - A_1(y))^2}{e^{-2z}} + \frac{1}{e^{-2z}}\left(\frac{dS_y}{dy}\right)^2 + \left(\frac{dS_z}{dz}\right)^2 - \left(\frac{dS_\eta}{d\eta}\right)^2 - M^2\alpha(\eta)^2 = 0. \tag{10}$$

4

Equation (10) reduces to the following system of differential equations:

$$\left(\frac{dS_z}{dz}\right)^2 + k_{xy}^2 e^{2z} = k_z^2,$$ (11)

$$\left(\frac{dS_\eta}{d\eta}\right)^2 + M^2\alpha(\eta)^2 = k_z^2,$$ (12)

$$(k_x - A_1(y))^2 + \left(\frac{dS_y}{dy}\right)^2 = k_{xy}^2,$$ (13)

where $k_{xy}^2$, and $k_z^2$ are separation constants.

In the absence of electromagnetic interaction, we have that $A_1(y) = 0$ and the solution of Eq. (13) takes the form

$$S_y = \pm\sqrt{k_{xy}^2 - k_x^2}\, y = \pm k_y y.$$ (14)

where we have introduced the constant $k_y$. Equation (14) can also be derived looking at the symmetry between the torus coordinates $x$ and $y$ in the line element (3) and Eq. (10) when $A_1(y) = 0$.

When the vector potential has the simple form $A_1(y) = A_1 y$, the magnetic field reads $B = \frac{e^{2z}}{\alpha(\eta)^2}\,|A_1|$ and the function $S_y(y)$ is

$$S_y(y) = -\frac{k_x - A_1 y}{2A_1}\sqrt{k_{xy}^2 - (k_x - A_1 y)^2} + \frac{k_{xy}^2}{2A_1}\arctan\frac{A_y y - k_x}{\sqrt{k_{xy}^2 - (k_x - A_1 y)^2}},$$ (15)

The solution of Eq. (11) can be expressed in terms of elementary functions as follows:

$$S_z = \sqrt{k_z^2 - k_{xy}^2\exp(2z)} - k_z\tanh^{-1}\sqrt{\frac{k_z^2 - k_{xy}^2\exp(2z)}{k_z^2}}.$$ (16)

The solution of Eq. (12) can be written as

$$S_\eta = \pm\int\sqrt{k_z^2 - M^2\alpha(\eta)^2}\,d\eta,$$ (17)

whose explicit form in terms of elementary functions will depend on a particular choice of the expansion function $\alpha(\eta)$.

Since we have been able to solve the Hamilton-Jacobi equation in the Stäckel space given by (4), we can construct the quasiclassical modes of the relativistic wave equations through the identification

$$\Phi \to e^{iS} = e^{\pm i \int \sqrt{k_z^2 - M^2 \alpha(\eta)^2} d\eta} e^{ik_x x} e^{iS_y} e^{iS_z}, \tag{18}$$

where $S_z$ and $S_y$ take the following values at the asymptotes

$$S_{z(\infty)} \to i k_{xy} e^z, \ S_{z(-\infty)} \to k_z z, \tag{19}$$

$$S_{y(\infty)} \to \mp i \frac{(k_x - A_1 y)^2}{2A_1}. \tag{20}$$

When the electromagnetic interaction is not present we have that $S_y = \exp(ik_y y)$.

### III. SOLUTION OF THE KLEIN GORDON EQUATION

The covariant generalization of the Klein Gordon equation in curved space-time has the form [12]

$$g^{\alpha\beta} \left( \nabla_\alpha - ieA_\alpha \right) \left( \nabla_\beta - ieA_\beta \right) \Phi - (M^2 + \xi R)\Phi = 0, \tag{21}$$

where

$\nabla_\alpha$ is the covariant derivative, $R$ is the curvature scalar and $\xi$ is a scalar dimensionless coupling constant which takes the value $\xi = 1/6$ in the conformal case and $\xi = 0$ when a minimal coupling is considered. The value of the $R$ for the metric (3) is

$$R = 6 \frac{-a(\eta) + \frac{d^2 a(\eta)}{d\eta^2}}{a(\eta)^3}. \tag{22}$$

Substituting the metric associated with the line element (3) into the Klein Gordon equation (21) one obtains

$$e^{2z} \frac{\partial^2 \Phi}{\partial x^2} + e^{2z} \frac{\partial^2 \Phi}{\partial y^2} - 2 \frac{\partial \Phi}{\partial z} + \frac{\partial^2 \Phi}{\partial z^2} - 2 \frac{\partial \Phi}{\partial \eta} \frac{d\alpha}{d\eta} \frac{1}{\alpha^3} - e^{2z} A_1(y)^2 \Phi - 2ie^{2z} \frac{\partial \Phi}{\partial x} A_1(y) - M^2 \alpha^2 \Phi = 0, \tag{23}$$

6

where we have chosen to work with a minimal coupling $\xi = 0$. The Klein Gordon equation (21) is completely separable in (3), therefore we look for its solution in the form.

$$\Phi = H(\eta)Z(z)Y(y)e^{ik_x x}. \tag{24}$$

Substituting (24) into Eq. (21) we reduce the problem of solving the Klein-Gordon equation to that of finding solutions of the following set of ordinary differential equations

$$\frac{d^2Y}{dy^2} - \left((k_x - A_1(y))^2 - k^2\right)Y = 0, \tag{25}$$

$$\frac{d^2Z}{dz^2} - 2\frac{dZ}{dz} - (\lambda^2 + k^2 e^{2z})Z = 0, \tag{26}$$

$$\frac{d^2H}{d\eta^2} + 2\frac{dH}{d\eta}\frac{d\alpha}{d\eta} + (\alpha^2(\eta)M^2 - \lambda^2)H = 0, \tag{27}$$

with $\lambda^2$ and $k^2$ as separation constants. For $A(y) = A_1 y$ the solution of Eq. (25) can be expressed in terms of Whittaker functions [27] as follows

$$Y = C_1 v^{-1/2} M_{\frac{k^2}{4A_1},\frac{1}{4}}(v^2) + C_2 v^{-1/2} W_{\frac{k^2}{4A_1},\frac{1}{4}}(v^2), \tag{28}$$

where

$$v = \frac{A_1 y - k_x}{\sqrt{A_1}}, \tag{29}$$

and $C_1$ and $C_2$ are arbitrary constants. In the absence of electromagnetic field the solution of Eq. (25) reduces to

$$Y = C_1 e^{\pm i\sqrt{k^2 - k_x^2}y} = C_1 e^{\pm ik_y y}. \tag{30}$$

The solution of Eq. (26) is [28]

$$Z = C_3 e^z H^{(1)}_{\sqrt{1+\lambda^2}}(ike^z) + C_4 e^z H^{(2)}_{\sqrt{1+\lambda^2}}(ike^z), \tag{31}$$

where $H^{(1)}_\nu(z)$ and $H^{(2)}_\nu(z)$ are the Hankel functions and $C_3$ and $C_4$ are arbitrary constants. We can also express the solution of (26) in terms of Bessel functions $J_\nu(z)$ as

$$Z = D_3 e^z J_{\sqrt{1+\lambda^2}}(ike^z) + D_4 e^z J_{-\sqrt{1+\lambda^2}}(ike^z) \tag{32}$$

where $D_3$ and $D_4$ are arbitrary constants.

After introducing the function $h(\eta)$:

$$H(\eta) = \frac{h(\eta)}{\alpha(\eta)}, \tag{33}$$

Eq. (27) reduces to

$$\frac{d^2h}{d\eta^2} + (\alpha^2(\eta)M^2 - \lambda^2 - \frac{d^2\alpha(\eta)}{d\eta^2})h = 0. \tag{34}$$

In order to analyze the asymptotic behavior of the solutions of the Klein-Gordon equation (21) we make use of the asymptotic behavior of the Hankel functions [27]

$$H_\nu^{(1)}(z) \to \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-\pi\nu/2-\pi/4)}, H_\nu^{(2)}(z) \to \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z-\pi\nu/2-\pi/4)}, \tag{35}$$

as $z \to \infty$, and the behavior of $J_\nu(z)$ as $z \to 0$ [28]

$$J_\nu(z) \to \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(1+\nu)} \tag{36}$$

The asymptotic behavior of the Whittaker function $W_{k,\mu}(z)$ for large values of $z$ is [28]

$$W_{k,\mu}(z) \to e^{-z/2} z^k, \tag{37}$$

and the function $M_{k,\mu}(z)$ has the following asymptotic behavior as $z \to 0$

$$M_{k,\mu}(z) \to e^{-z/2} z^{\mu+\frac{1}{2}}. \tag{38}$$

An approximate solution of Eq. (34) can be obtained provided that the expansion parameter $\alpha(\eta)$ satisfies the conditions of validity of the adiabatic approximation. In this case one has that $h(\eta)$ has the form [29–31]

$$h(\eta) = \frac{1}{\sqrt{2W(\eta)}} \exp(\pm i \int^\eta W(\eta')d\eta', \tag{39}$$

with

$$W(\eta)^2 = \omega(\eta)^2 \left[1 + \delta_2(\eta)\omega^{-2} + ...\right], \tag{40}$$

8

where the function $\omega(\eta)$ has the form

$$\omega(\eta)^2 = \alpha^2(\eta)M^2 - \lambda^2 - \frac{d^2\alpha(\eta)}{d\eta^2}, \tag{41}$$

$\delta_n(\eta)$ is a function of $\omega(\eta)$ and its derivatives at $\eta$ up through $\omega^{(n)}(\eta)$ and $\delta_n(\eta)$ is bounded as $\omega \to \infty$. The solution of the Klein-Gordon equation (21) can be written as

$$\Phi = \frac{\exp(\pm i \int \sqrt{\alpha^2(\eta)M^2 - \lambda^2 - \frac{d^2\alpha(\eta)}{d\eta^2}}d\eta}{\sqrt{2}a(\eta)^{3/2}}Z(z)Y(y)e^{ik_x x}. \tag{42}$$

Let us analyze the asymptotic behavior of (42) as $y \to \infty$ and $z \to -\infty$. Using (19) and (36) we obtain that, when the electromagnetic interaction is switched off, the mode solutions of Eq. (21) take the asymptotic form

$$\Phi \to \frac{\exp(\pm i \int \sqrt{\alpha^2(\eta)M^2 - \lambda^2 - \frac{d^2\alpha(\eta)}{d\eta^2}}d\eta}{\sqrt{2}a(\eta)^{3/2}}e^{\mp ik_y y}e^{ik_x x}e^{(\mp\sqrt{1+\lambda^2}+1)z}. \tag{43}$$

Analogously, we have that in the presence of the electromagnetic potential the mode solutions of Eq. (21) take the following asymptotic form

$$\Phi \to \frac{\exp(\pm i \int \sqrt{\alpha^2(\eta)M^2 - \lambda^2 - \frac{d^2\alpha(\eta)}{d\eta^2}}d\eta}{\sqrt{2}a(\eta)^{3/2}}e^{(\pm\sqrt{1+\lambda^2}+1)z}v^{k^2/2A_1}e^{-v^2/2}e^{ik_x x}. \tag{44}$$

For large positive values of $z$ we have that the asymptotic behavior of $\Phi$ is

$$\Phi \to \frac{\exp(\pm i \int \sqrt{\alpha^2(\eta)M^2 - \lambda^2 - \frac{d^2\alpha(\eta)}{d\eta^2}}d\eta}{\sqrt{2}a(\eta)^{3/2}}e^{\mp e^{-ke^z}}e^{z/2}v^{k^2/2A_1}e^{-v^2/2}e^{ik_x x}. \tag{45}$$

From Eq. (44) we can identify the quasiclassical modes as $y \to \infty$ and $z \to -\infty$ as

$$\Phi_{class(z\to-\infty)} = \frac{h(\eta)}{\alpha(\eta)}e^z J_{\pm\sqrt{1+\lambda^2}}(ike^z)v^{-1/2}W_{\frac{k^2}{4A_1},\frac{1}{4}}(v^2)e^{ik_x x}. \tag{46}$$

Analogously, from Eq. (45) we have that the quasiclassical modes as $y \to \infty$ and $z \to \infty$ are

$$\Phi_{class(z\to\infty)} = \frac{h(\eta)}{\alpha(\eta)}e^z H^{(1,2)}_{\pm\sqrt{1+\lambda^2}}(ike^z)v^{-1/2}W_{\frac{k^2}{4A_1},\frac{1}{4}}(v^2)e^{ik_x x}. \tag{47}$$

9

## IV. SOLUTION OF THE DIRAC EQUATION

The Dirac equation is a system of coupled partial differential equations which is separable in a very restricted set of metrics. Among the spacetimes where the separability of the Klein-Gordon and Dirac equations has been studied one can mention the Stäckel spaces [26], which are those metrics where the Hamilton-Jacobi equation is separable. Nevertheless recently it has been shown that this condition is neither necessary nor sufficient in order to guarantee a complete separability of variables in the Dirac equation (see Ref. [32] and references therein). A systematic classification of the gravitational backgrounds where the Dirac equation is separable with the help of the algebraic method is presented in ref. [20]. The line element (3) belongs to this family and consequently one can apply the algebraic method of separation.

The covariant generalization of the Dirac equation in curved space-time is [12,33]

$$\tilde{\gamma}^\alpha \left( \partial_\alpha - \Gamma_\alpha - ieA_\alpha \right) \tilde{\Psi} + M\tilde{\Psi} = 0, \tag{48}$$

where the curved Dirac matrices $\tilde{\gamma}^\alpha$ satisfy the commutation relation

$$\left\{ \tilde{\gamma}^\alpha, \tilde{\gamma}^\beta \right\} = 2g^{\alpha\beta}, \tag{49}$$

and $\Gamma_\alpha$ are the spin connections [33]

$$\Gamma_\alpha = \frac{1}{4} g_{\mu\lambda} \left[ \left( \frac{\partial b_\nu^\beta}{\partial x^\alpha} \right) a_\beta^\lambda - \Gamma_{\nu\alpha}^\lambda \right] s^{\mu\nu}, \tag{50}$$

where

$$s^{\mu\nu} = \frac{1}{2} \left( \tilde{\gamma}^\mu \tilde{\gamma}^\nu - \tilde{\gamma}^\nu \tilde{\gamma}^\mu \right), \tag{51}$$

and the matrices $b_\mu^\alpha$, $a_\beta^\mu$ establish the connection between the Dirac matrices $\tilde{\gamma}^\mu$ on a curved space-time and the flat Dirac matrices $\gamma^\mu$ as follows:

$$\tilde{\gamma}_\mu = b_\mu^\alpha \gamma_\alpha, \quad \tilde{\gamma}^\mu = a_\beta^\mu \gamma^\beta. \tag{52}$$

Since the line element (3) is associated with a diagonal metric, we can work in the diagonal tetrad gauge for $\tilde{\gamma}^\mu$:

$$\tilde{\gamma}^0 = \frac{\gamma^0}{a(\eta)}, \quad \tilde{\gamma}^1 = \frac{\gamma^1}{a(\eta)e^{-z}}, \quad \tilde{\gamma}^2 = \frac{\gamma^2}{a(\eta)e^{-z}}, \quad \tilde{\gamma}^3 = \frac{\gamma^0}{a(\eta)}, \tag{53}$$

substituting (53) into (50) we obtain that the spinor connections are

$$\Gamma_1 = -\frac{1}{2}\frac{e^{-z}}{\alpha(\eta)}\left\{-\alpha(\eta)\gamma^1\gamma^3 + \frac{d\alpha(\eta)}{d\eta}\gamma^1\gamma^4\right\}, \tag{54}$$

$$\Gamma_2 = -\frac{1}{2}\frac{e^{-z}}{\alpha(\eta)}\left\{-\alpha(\eta)\gamma^2\gamma^3 + \frac{d\alpha(\eta)}{d\eta}\gamma^2\gamma^4\right\}, \tag{55}$$

$$\Gamma_3 = -\frac{1}{2}\frac{d\alpha(\eta)}{d\eta}\frac{1}{\alpha(\eta)}\gamma^3\gamma^4, \quad \Gamma_4 = 0. \tag{56}$$

Substituting (53)-(56) into (48) we find that the Dirac equation takes the simple form

$$\left\{\gamma^0\frac{\partial}{\partial\eta} + \gamma^1 e^z\left(\frac{\partial}{\partial x} - A_1(y)\right) + \gamma^2 e^z\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z} + M\alpha(\eta)\right\}\Psi = 0, \tag{57}$$

where we have introduced the spinor $\Psi$

$$\tilde{\Psi} = a(\eta)^{-3/2}e^z\Psi. \tag{58}$$

Regarding Eq. (57) we should mention that it does exhibits a nonfactorizable structure [22,34]. In order to solve Eq. (57) we apply the algebraic method of separation of variables [20–24]. The method consists in rewriting the Dirac equation (57) as a sum of two first order differential operators $\hat{K}_1, \hat{K}_2$ satisfying the relation

$$\left[\hat{K}_1, \hat{K}_2\right]_- = 0, \quad \left\{\hat{K}_1 + \hat{K}_2\right\}\Phi = 0 \tag{59}$$

with

$$\gamma^3\gamma^0\Psi = \Phi, \tag{60}$$

and

$$\hat{K}_1(x, y)\Phi = \left\{\gamma^2\frac{\partial}{\partial y} + \gamma^1\left(\frac{\partial}{\partial x} - iA_1(y)\right)\right\}\gamma^3\gamma^0\Phi = ik\Phi, \tag{61}$$

$$\hat{K}_2(z, \eta)\Phi = e^z\left\{\gamma^0\frac{\partial}{\partial\eta} + \gamma^3\frac{\partial}{\partial z} + M\alpha(\eta)\right\}\gamma^3\gamma^0\Phi = -ik\Phi. \tag{62}$$

11

It should be noticed that using the pairwise scheme of separation one has been able to reduce the problem of solving the Dirac equation to finding solutions of the decoupled system of Eqs. (61) and (62). A further problem arises when we try to separate variables in Eq. (62). Here it is not possible to reduce the problem to a set of two commuting first order differential operators. In order to separate variables in Eq. (62) we re-write it in the following form: [22,35]

$$\left( \hat{L}_1 \gamma^3 \gamma^0 + \hat{L}_2 \right) \Phi = 0, \tag{63}$$

where $\hat{L}_1$ and $\hat{L}_2$ are two commuting differential operators given by the expressions

$$\hat{L}_1 = \gamma^0 \frac{\partial}{\partial \eta} + M\alpha(\eta), \tag{64}$$

$$\hat{L}_2 = \gamma^0 \frac{\partial}{\partial z} + ike^z, \tag{65}$$

In order to separate variables in Eq. (63) we introduce the auxiliary spinor $\mathcal{Y}$

$$\left( \hat{L}_1 \gamma^3 \gamma^0 + \tilde{L}_2 \right) \mathcal{Y} = \Phi, \tag{66}$$

where the differential operator $\tilde{L}_2$ is given by the expression

$$\tilde{L}_2 = \gamma^0 \frac{\partial}{\partial z} - ike^z. \tag{67}$$

Substituting (66) into (63) we obtain that $\mathcal{Y}$ satisfies the following equation

$$\left\{ \hat{M}_1 + \hat{M}_2 \right\} \mathcal{Y} = 0, \tag{68}$$

with $\left[ \hat{M}_1, \hat{M}_2 \right] = 0$, and

$$\left( \hat{M}_1 + \tilde{\lambda} \right) \mathcal{Y} = \left( -\frac{\partial^2}{\partial z^2} - i\gamma^0 ke^z + k^2 e^{2z} + \tilde{\lambda} \right) \mathcal{Y} = 0, \tag{69}$$

$$\left( \hat{M}_2 - \tilde{\lambda} \right) \mathcal{Y} = \left( \frac{\partial^2}{\partial \eta^2} + \gamma^0 M \frac{d\alpha(\eta)}{d\eta} + M^2 \alpha^2(\eta) - \tilde{\lambda} \right) \mathcal{Y} = 0, \tag{70}$$

where $\tilde{\lambda}$ is a separation constant. Introducing the new variable $u = 2ke^z$, we have that Eq. (69) can be written as

$$\left( \frac{\partial^2}{\partial u^2} + \frac{i}{2u}\gamma^0 - \frac{1}{4} + \left(\frac{1}{4} - \lambda\right)\frac{1}{u^2} \right)\mathcal{S} = 0, \tag{71}$$

where

$$u^{-1/2}\mathcal{S} = \mathcal{Y}. \tag{72}$$

Choosing the following representation of the Dirac matrices [36]

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \quad 1 \le j \le 3 \tag{73}$$

we readily obtain that the spinor $\Phi$ has the following structure

$$\left[ \sigma_1 \frac{\partial}{\partial y} - i\sigma_2(k_x - A_1(y)) \right]\Phi_1 = ik\Phi_2, \tag{74}$$

$$\left[ -\sigma_1 \frac{\partial}{\partial y} + i\sigma_2(k_x - A_1(y)) \right]\Phi_2 = ik\Phi_1, \tag{75}$$

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \phi(y) \\ F\sigma^3\phi(y) \end{pmatrix} \exp(ik_x x), \tag{76}$$

where

$$\phi(y) = \begin{pmatrix} A(y) \\ B(y) \end{pmatrix}. \tag{77}$$

Using the representation (73) we obtain that the solution of Eq. (71) can be written in terms of Whittaker functions

$$S_{1,2} = D_1 W_{-1/2,\sqrt{\lambda}}(u) + D_2 M_{-1/2,\sqrt{\lambda}}(u), S_{3,4} = D_3 W_{1/2,\sqrt{\lambda}}(u) + D_4 M_{1/2,\sqrt{\lambda}}(u) \tag{78}$$

where $D_1$, $D_2$, $D_3$, $D_4$ do not depend on the variable $u$. Looking at (70) and (71) we have that, for regular solutions at $u = 0$, the spinor $\mathcal{Y}$ has the following structure:

$$\mathcal{Y} = \begin{pmatrix} a(y)c_1(\eta)u^{-1/2}M_{+\frac{1}{2},\sqrt{\lambda}}(u) \\ b(y)c_1(\eta)u^{-1/2}M_{+\frac{1}{2},\sqrt{\lambda}}(u) \\ c(y)c_2(\eta)u^{-1/2}M_{-\frac{1}{2},\sqrt{\lambda}}(u) \\ d(y)c_2(\eta)u^{-1/2}M_{-\frac{1}{2},\sqrt{\lambda}}(u) \end{pmatrix} \exp(ik_x x), \tag{79}$$

13

Substituting (79) into (66) and noticing that Eq. (70) is equivalent to the following system of equations

$$\left(\frac{\partial}{\partial\eta} - iM\alpha(\eta)\right)c_1(\eta) = \sqrt{\tilde{\lambda}}c_2(\eta), \tag{80}$$

$$\left(\frac{\partial}{\partial\eta} + iM\alpha(\eta)\right)c_2(\eta) = \sqrt{\tilde{\lambda}}c_1(\eta), \tag{81}$$

we obtain that the spinor $\Phi$ has the following structure

$$\Phi = \begin{pmatrix} A(v)c_1(\eta)e^{-z/2}M_{-\frac{1}{2},\sqrt{\tilde{\lambda}}}(2ke^z) \\ B(v)c_1(\eta)e^{-z/2}M_{-\frac{1}{2},\sqrt{\tilde{\lambda}}}(2ke^z) \\ iA(v)c_2(\eta)e^{-z/2}M_{\frac{1}{2},\sqrt{\tilde{\lambda}}}(2ke^z) \\ -iB(v)c_2(\eta)e^{-z/2}M_{\frac{1}{2},\sqrt{\tilde{\lambda}}}(2ke^z) \end{pmatrix}\exp(ik_x x), \tag{82}$$

where $A(v)$ and $B(v)$ satisfy the system coupled system of equations

$$\left(\frac{d}{dy} - (k_x - A_1(y))\right)B = ikA, \tag{83}$$

$$\left(\frac{d}{dy} + (k_x - A_1(y))\right)A = ikB, \tag{84}$$

where $v$ was defined in Eq. (29).

The corresponding solution of Eq. (59) in terms of the Whittaker functions $W_{k,\mu}(z)$ has the form

$$\Phi = \begin{pmatrix} \sqrt{\tilde{\lambda}}iA(v)c_1(\eta)e^{-z/2}W_{-\frac{1}{2},\sqrt{\tilde{\lambda}}}(2ke^z) \\ -i\sqrt{\tilde{\lambda}}B(v)c_1(\eta)e^{-z/2}W_{-\frac{1}{2},\sqrt{\tilde{\lambda}}}(2ke^z) \\ A(v)c_2(\eta)e^{-z/2}W_{\frac{1}{2},\sqrt{\tilde{\lambda}}}(2ke^z) \\ B(v)c_2(\eta)e^{-z/2}W_{\frac{1}{2},\sqrt{\tilde{\lambda}}}(2ke^z) \end{pmatrix}\exp(ik_x x). \tag{85}$$

Let us look for solutions of the system (83) and (84) when the electromagnetic potential has the simple functional dependence $A_1(y) = A_1y$. In this case one can obtain exact solutions

14

for $A(v)$ and $B(v)$ in terms of hypergeometric functions. After making the change of variable (29) and using the recurrence relations [27]

$$(b-1)M(a, b-1, z) = (b-1)M(a, b, z) + z\frac{dM(a, b, z)}{dz}, \tag{86}$$

$$\frac{1}{a}\frac{dM(a, b, z)}{dz} + M(a, b, z) = M(a+1, b, z), \tag{87}$$

$$\frac{dU(a, b, z)}{dz} - U(a, b, z) = -U(a, b+1, z), \tag{88}$$

we find that the general solution of the system of equations (83) and (84) reads

$$A = \frac{\sqrt{2A_1}}{ik}e^{-\frac{1}{2}v^2}(C_1 M(-\frac{k^2}{4A_1} + \frac{1}{2}, \frac{1}{2}, v^2) + C_2 U(-\frac{k^2}{4A_1} + \frac{1}{2}, \frac{1}{2}, v^2)), \tag{89}$$

$$B = e^{-\frac{1}{2}v^2}v(C_1 M(-\frac{k^2}{4A_1} + \frac{1}{2}, \frac{3}{2}, v^2) - C_2 U(-\frac{k^2}{4A_1} + \frac{1}{2}, \frac{3}{2}, v^2)). \tag{90}$$

The exact solution of the system of equations (83)-(84) in the absence of electromagnetic interaction has the form

$$A = C_1 e^{i\sqrt{k^2-k_x^2}y} + C_2 e^{-i\sqrt{k^2-k_x^2}y}, \tag{91}$$

$$B = \frac{\sqrt{k^2-k_x^2}-ik_x}{k}C_1 e^{i\sqrt{k^2-k_x^2}y} - \frac{\sqrt{k^2-k_x^2}+ik_x}{k}C_2 e^{-i\sqrt{k^2-k_x^2}y}, \tag{92}$$

where $C_1$ and $C_2$ are arbitrary constants. The solutions of the Dirac equation (82) and (85) exhibit an asymptotic behavior which can be identified with the quasiclassical solutions of the Hamilton-Jacobi equation (4). With the help of the asymptotic expressions (38), we find that the Dirac spinor $\Phi$ as $z \to -\infty$, and $y \to \infty$. takes the form

$$\Phi_{z\to-\infty} = \begin{pmatrix} \frac{\sqrt{2A_1}}{ik}c_1(\eta) \\ -vc_1(\eta) \\ -\frac{\sqrt{2A_1}}{k}c_2(\eta) \\ ivc_2(\eta) \end{pmatrix} e^{-ke^z}e^{\sqrt{\tilde{\lambda}}z}e^{-\frac{v^2}{2}}v^{\frac{k^2}{2A_1}-1}\exp(ik_x x), \tag{93}$$

where the functions $c_1(\eta)$ and $c_2(\eta)$ satisfy the system of equations (80) and (81). For asymptotically large values of $z$ we have that the spinor $\Phi$ takes the form.

$$\Phi_{z\to\infty} = \begin{pmatrix} \sqrt{\tilde{\lambda}}\frac{\sqrt{2A_1}}{k}c_1(\eta)e^{-z} \\ i\sqrt{\tilde{\lambda}}vc_1(\eta)e^{-z} \\ -i2\sqrt{2A_1}c_2(\eta) \\ -2kvc_2(\eta) \end{pmatrix} e^{-ke^z}e^{-\frac{v^2}{2}}v^{\frac{k^2}{2A_1}}\exp(ik_xx). \tag{94}$$

Looking at the solution of the Hamilton-Jacobi equation we can identify (82) and (85) as the corresponding quasiclassical modes as $z \to -\infty$ and $z \to \infty$, respectively. An approximate expression for the time dependence of the spinor $\Phi$ can be obtained with the help of the WKB approximation. In this case we obtain

$$c_2(\eta) \sim c_{10}\exp(i\omega(\eta)), \tag{95}$$

$$c_1(\eta) \sim -i\frac{c_{10}}{\omega(\eta) + M\alpha(\eta)}\exp(i\omega(\eta)), \tag{96}$$

where $c_{10}$ is a normalization constant and $\omega(\eta) = \sqrt{iM\frac{d\alpha}{d\eta} + M^2\alpha^2 - \tilde{\lambda}}$. Looking at (95)-(96) and (93) we readily see that, for large values of $\eta$ we obtain $c_1(\eta) \to -i\frac{c_{10}}{2M\alpha(\eta)}\exp(i\omega(\eta))$. Analytic solutions of the system of equations (95) and (96) can be obtained for some particular expansion parameter $\alpha(\eta)$ [22,35]

## V. CONCLUDING REMARKS

In this article, we have solved the Klein-Gordon and Dirac equations in an open cosmological universe with partially horn topology. The solutions of the relativistic wave equations are expressed in terms of special functions. In Sec. IV we have shown that the algebraic method of separation [22–24,35] permits one a complete separation of variables of the Dirac equation in the line element associated with a horn topology. The identification of the quasiclassical modes with the help of the relativistic Hamilton-Jacobi equation shows that this method is a very useful tool in the study of quantum effects in curved spaces.

16

As a final remark, we should mention that the introduction of nonstandard topologies in order to describe the large scale structure of the space-time also opens new possibilities to discuss quantum effects in globally inhomogeneous and anisotropic backgrounds in the presence of non-trivial electromagnetic interactions.

## ACKNOWLEDGMENTS

# REFERENCES

[1] P. de Bernardis et al. Proc. CAPP2000 (Verbier, 17-28 July 2000) (Preprint astro-ph/0011469).

[2] S. Perlmutter et al, Astrophys J. **517**, 565 (1999).

[3] S. Perlmutter, M. S. Turner, and M. White, Phys. Rev. Lett. **83**, 670 (1999).

[4] J. Levin, J. Barrow, E. F. Bunn, and J. Silk, Phys. Rev. Lett. **79**, 974 (1997).

[5] D. D. Sokolov and A. A. Starobinskii, Sov. Astron. **19**, 629 (1976).

[6] G. I. Gomero, M. J. Rebouças, and R. Tavakol, Class. Quantum Grav. **18**, 4461 (2001).

[7] G. I. Gomero, M. J. Rebouças, and A. F. F. Texeira, Int. J. Mod. Phys. D. **9** 678 (2000).

[8] G. I. Gomero, M. J. Rebouças, and R. Tavakol, Class. Quantum Grav. **18** L145 (2001).

[9] A. L. Zelmanov, Dokl. AN. URSS **124**, 1030 (1959).

[10] Yu. S. Vladimirov *Sistemy otscheta v teorii gravitatsii* (Enerogoizdat, Moscow, 1982).

[11] E. Schrödinger, Physica (Amsterdam) **6**, 899 (1939); Proc. R. Ir. Acad. Sci., Sect. A. **46**, 9 (1940).

[12] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved* (Cambridge University Press, Cambridge, 1982).

[13] A. A. Grib, S. G. Mamaev and V. M. Mostepanenko *Quantum Vacuum effects in strong fields* (Energoatomizdat, Moscow, 1988).

[14] L. Parker, Phys. Rev. Lett. **21** , 562 (1968), *Asymptotic Structure of Space-Time* (Plenum, New York, 1977).

[15] A. O. Barut and I. H. Duru, Phys. Rev. D. **36**, 3705 (1987).

[16] S.P. Gavrilov, D.M. Gitman, and A.E. Goncalves, Int.J.Mod.Phys. A **16**, 4235 (2001).

[17] L. Parker, Phys. Rev. D. **3**, 346 (1971).

[18] V. G. Bagrov, and D. M. Gitman *Exact Solutions of Relativistic Wave Equations* (Kluwer, Dordrecht, 1990).

[19] V. G. Bagrov, A. B. Shapovalov, and A. A. Yevseyevich, Class. Quantum. Grav. **8**, 163 (1991).

[20] I. E. Andrushkevich, and G. V. Shishkin, Theor. Math. Phys. 70, **204** (1987).

[21] G. V. Shishkin, and V.M. Villalba, J. Math. Phys. **30**, 2132 (1989).

[22] V. M. Villalba, Mod. Phys. Lett A. **8**, 2351 (1993).

[23] M. N. Hounkonnou, and J. E. B. Mendy, J. Math. Phys **40,** 3827 (1999).

[24] M. N. Hounkonnou, and J. E. B. Mendy, J. Math. Phys **40**, 4240 (1999).

[25] L. D. Landau, and E. M. Lifshitz, *The Classical Theory of Fields* (Addison Wesley, Reading MA, 1973).

[26] P. Stäckel, Math. Ann. **42**, 537 (1893).

[27] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York 1974).

[28] N. N. Lebedev *Special functions and their applications* (Dover, New York 1972).

[29] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, Cambridge, 1991).

[30] L. Parker, Phys. Rev. **183**, 1057 (1969).

[31] L. Parker and S. Fulling, Phys. Rev. D. **9**, 341 (1974).

[32] O. L.Varaksin, and I. V. Shirokov, Izvestya. Vuzov, Fizika **1**, 31 (1996).

[33] D. Brill, and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957).

[34] M. Fels and N. Kamran , Proc. R. Soc. London A **425**, 229 (1990).

[35] V. M. Villalba, and U. Percoco, J. Math. Phys. **31**, 715 (1990).

[36] J.M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Springer-Verlag, New York, 1976).